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Chaotic and hyperchaotic attractors of a complex nonlinear system

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Abstract
In this paper, we introduce a complex nonlinear hyperchaotic system which is a five-dimensional system of nonlinear autonomous differential equations. This system exhibits both chaotic and hyperchaotic behavior and its dynamics is very rich. Based on the Lyapunov exponents, the parameter values at which this system has chaotic, hyperchaotic attractors, periodic and quasi-periodic solutions and solutions that approach fixed points are calculated. The stability analysis of these fixed points is carried out. The fractional Lyapunov dimension of both chaotic and hyperchaotic attractors is calculated. Some figures are presented to show our results. Hyperchaos synchronization is studied analytically as well as numerically, and excellent agreement is found.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In 1982, Fowler *et al* [1] introduced the complex Lorenz model as

$$\dot{x} = a(y - x), \qquad \dot{y} = cx - y - xz, \qquad \dot{z} = -bz + \tfrac{1}{2}(\bar{x}y + x\bar{y}), \qquad (1.1)$$

where x and y are complex functions, z is a real function and a, b and c are positive parameters. Dots represent derivatives with respect to time and an overbar denotes the complex conjugate function. Equations (1.1) describe and simulate the physics of detuned lasers [2 and references therein]. The functions x, y and z are related to the electric field, the atomic polarization amplitudes and the population inversion, respectively; for more details, see [2]. Recently, the basic properties and chaos synchronization of model (1.1) have been studied [3]. It is shown that the complex Lorenz model is chaotic and has only chaotic attractors. Chaos is found to be

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useful and has great potential applications in many important fields such as communications, rotating fluids dynamics and biomedical engineering applications to the human brain and heart [4–6]. Other complex nonlinear systems have been introduced and studied in recent years [7–9]. The knowledge of the dynamics of complex systems like (1.1) is still far from that achieved for their real counterparts [7–13].

We define chaos as sensitive dependence on initial conditions. A dynamical system is defined as a hyperchaotic system if it has at least two positive Lyapunov exponents. Hyperchaotic systems exhibit more complex nonlinear behavior. These systems with real variables (or functions) have been introduced and studied in the literature [14–16].

In this work we wish to construct a new system, which is a hyperchaotic one, by adding the cross-product nonlinear term yz to the first equation of (1.1) as follows:

$$\dot{x} = a(y - x) + yz, \qquad \dot{y} = cx - y - xz, \qquad \dot{z} = -bz + \tfrac{1}{2}(\bar{x}y + x\bar{y}), \tag{1.2}$$

where $x = v_1 + \mathrm{i}v_2$, $y = v_3 + \mathrm{i}v_4$ are complex functions (variables), $\mathrm{i} = \sqrt{-1}$, $z = v_5$ and v_i, $i = 1, \ldots, 5$, are real functions.

In 2005, the real counterpart of (1.2) (i.e. x, y and z are real functions) was introduced and studied in [4], where it was shown that it does not have hyperchaotic attractors. The dynamics of (1.2) is more rich in the sense that it exhibits both chaotic and hyperchaotic attractors, while the complex Lorenz model (1.1), complex Chen and Lü systems all have only chaotic attractors [3, 7]. In the literature, it has been reported that hyperchaotic attractors can be generated by adding a state feedback control to the three-dimensional real nonlinear systems [6, 14–16].

This paper is organized as follows. In section 2 we study the dynamical properties of our nonlinear hyperchaotic system (1.2), showing that it possesses an isolated fixed point E_0 at (0, 0, 0, 0, 0) as well as two whole circles of equilibria. The stability analysis of E_0 is carried out. The parameter values at which this system has chaotic and hyperchaotic attractors are calculated based on the Lyapunov exponents. The fractional Lyapunov dimension [17, 18] of these attractors is also obtained. A distinctive feature of the chaotic and hyperchaotic attractors is their fractional (noninteger) dimension. It is also shown that (1.2) has periodic, quasi-periodic solutions and solutions that approach fixed points. In section 3, the synchronization of hyperchaotic attractors is achieved using a nonlinear control method [19, 20]. Furthermore, the Lyapunov function is derived to show that the error states (i.e. differences in the dynamics of the two systems) converge to zero. A good agreement is found between analytical and numerical results. Finally, section 4 sums up the main conclusions of this work.

2. Complex dynamics

In this section we study the dynamical properties of (1.2), including the property of dissipation, existence of fixed points and their stability, observation of hyperchaotic and chaotic attractors and complex dynamical behaviors.

The real version of (1.2) is described by

$$\begin{aligned} &\dot{v}_1 = a(v_3 - v_1) + v_3 v_5, \qquad \dot{v}_2 = a(v_4 - v_2) + v_4 v_5, \qquad \dot{v}_3 = cv_1 - v_5 v_1 - v_3, \\ &\dot{v}_4 = cv_2 - v_5 v_2 - v_4, \qquad \dot{v}_5 = -bv_5 + (v_1 v_3 + v_2 v_4). \end{aligned} \tag{2.1}$$

System (1.2) (or (2.1)) has the following fundamental dynamical properties.

2.1. Dissipation

System (1.2) is dissipative under the condition $(2a + b + 2) > 0$, since

$$\frac{\partial \dot{v}_1}{\partial v_1} + \frac{\partial \dot{v}_2}{\partial v_2} + \cdots + \frac{\partial \dot{v}_5}{\partial v_5} = -(2a + b + 2). \tag{2.2}$$

2.2. Fixed points and their stability

The fixed points of (1.2) can be found by solving the equations

$$\begin{aligned} &a(v_3 - v_1) + v_3 v_5 = 0, \qquad a(v_4 - v_2) + v_4 v_5 = 0, \qquad c v_1 - v_5 v_1 - v_3 = 0,\\ &c v_2 - v_5 v_2 - v_4 = 0, \qquad -b v_5 + (v_1 v_3 + v_2 v_4) = 0. \end{aligned} \tag{2.3}$$

As a consequence, (1.2) possesses the following fixed points: an isolated one E_0 at $(0, 0, 0, 0, 0)$ and two whole circles of equilibria described by

$$v_1^2 + v_2^2 = r_1^2 \qquad \text{and} \qquad v_3^2 + v_4^2 = r_2^2, \tag{2.4}$$

where $r_1^2 = ab(1 - d)/d^2$ and $r_2^2 = d^2 r_1^2$, $d = (1/2)[(a + c) \pm \sqrt{(a + c)^2 - 4a}]$.

The nontrivial fixed points can be written in the form

$$E_\theta = \left(v_1, \pm\sqrt{(b/d)v_5 - v_1^2}, d v_1, \pm d\sqrt{(b/d)v_5 - v_1^2}, v_5\right), \tag{2.5}$$

where $v_5 = a(1 - d)/d$ and $v_1 = r_1 \cos\theta$, $v_2 = r_1 \sin\theta$, $v_3 = r_2 \cos\theta$, $v_4 = r_2 \sin\theta$, for $\theta \in [0, 2\pi]$.

To study the stability of $E_0 = (0, 0, 0, 0, 0)$, we calculate the Jacobian matrix of system (1.2) at E_0 as

$$J_{E_0} = \begin{pmatrix} -a & 0 & a & 0 & 0\\ 0 & -a & 0 & a & 0\\ c & 0 & -1 & 0 & 0\\ 0 & c & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -b \end{pmatrix},$$

and its eigenvalues satisfy the characteristic polynomial:

$$(\mu + b)\left[\mu^2 + (a + c)\mu + a(1 - c)\right]^2 = 0, \tag{2.6}$$

which yields $\mu_1 = -b < 0$, $\mu_{2,3} = -(1/2)[(a + c) - \sqrt{(a + c)^2 + 4a(c - 1)}]$ and $\mu_{4,5} = -(1/2)[(a+c)+\sqrt{(a + c)^2 + 4a(c - 1)}]$. Thus, for $c > 1$, we have $\mu_{2,3} > 0$, $\mu_{4,5} < 0$ and as a result E_0 is an unstable fixed point. The stability analysis of E_θ can be studied in a similar manner as for E_0.

2.3. Lyapunov exponents of (1.2)

In this subsection based on Lyapunov exponents, we calculate parameter values of (2.1) at which chaotic, hyperchaotic attractors, periodic, quasi-periodic solutions and solutions approaching fixed points exist.

System (2.1) in vector notation can be written as

$$\dot{V}(t) = h\,(V(t); \eta)\,, \tag{2.7}$$

where $V(t) = [v_1(t), v_2(t), v_3(t), v_4(t), v_5(t)]^t$ is the state space vector, $h = [h_1, h_2, h_3, h_4, h_5]^t$, η is a set of parameters and $[\ldots]^t$ denotes transpose. The equations for small deviations δV from the trajectory $V(t)$ are

$$\delta\dot{V}(t) = L_{ij}(V(t); \eta)\delta V, \qquad i, j = 1, 2, 3, 4, 5, \tag{2.8}$$

where $L_{i,j} = \frac{\partial h_i}{\partial v_j}$ is the Jacobian matrix:

$$L_{i,j} = \begin{pmatrix} -a & 0 & a+v_5 & 0 & v_3 \\ 0 & -a & 0 & a+v_5 & v_4 \\ c-v_5 & 0 & -1 & 0 & -v_1 \\ 0 & c-v_5 & 0 & -1 & -v_2 \\ v_3 & v_4 & v_1 & v_2 & -b \end{pmatrix}.$$

The Lyapunov exponents λ_i of the system are thus defined by [21]:

$$\lambda_i = \lim_{t\longrightarrow\infty} \frac{1}{t} \log \frac{\|\delta v_i(t)\|}{\|\delta v_i(0)\|}. \tag{2.9}$$

To find λ_i, equations (2.7) and (2.8) must be numerically solved simultaneously. For the case $a = 30, c = 90$ and $b = 11$ with the initial conditions $t_0 = 0; v_1(0) = 2$, $v_2(0) = 4, v_3(0) = 1, v_4(0) = 3$ and $v_5(0) = 2$ we calculate the Lyapunov exponents as $\lambda_1 = 5.333\,444, \lambda_2 = 0.282\,395, \lambda_3 = 0, \lambda_4 = -44.639\,339, \lambda_5 = -66.901\,647$.

This means that our system (2.1) for this choice of a, b and c is a hyperchaotic system since two of the Lyapunov exponents are positive and dissipative system since the sum of Lyapunov exponents is negative.

The solutions of system (2.1) can be classified using the sign of their associated Lyapunov exponents $\lambda_i, i = 1, 2, \ldots, 5$, as are shown in the following table:

λ_1	λ_2	λ_3	λ_4	λ_5	Type of solutions
–	–	–	–	–	Solutions approach fixed points
0	–	–	–	–	Periodic solutions (limit cycles)
0	0	–	–	–	Quasi-periodic solutions (2-torus)
+	0	–	–	–	Chaotic attractors
+	+	0	–	–	Hyperchaotic attractors
+	+	–	–	–	Hyperchaotic attractors

The Lyapunov dimension of the attractors of (1.2) is given by [17, 18]

$$D = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{2.10}$$

such that j is the largest integer for which $\sum_{i=1}^{j} \lambda_i > 0$.

2.3.1. Fix $c = 90, b = 11$ and vary a. For this case we calculate $\lambda_i, i = 1, 2, \ldots, 5$, for (1.2) from (2.9) with the initial conditions, $t_0 = 0; v_1(0) = 2, v_2(0) = 4, v_3(0) = 1$, $v_4(0) = 3$ and $v_5(0) = 2$. In figure 1(*a*) we plot λ_1, λ_2 and λ_3 versus a, while in figure 1(*b*) we plot λ_4 and λ_5 versus a. From these figures it is clear that (1.2) has hyperchaotic attractors for $a \in [(29.1, 30.1), (92.3, 102.1), (111.8, 123.9)]$, chaotic attractors for $a \in [(12, 12.8), (86.7, 87.1), (91.7, 92.3), (123.9, 149.6)]$, periodic attractors for $a \in [(0, 12], (110.5, 111.6), (149.6, 151)]$ and quasi-periodic solutions for $a \in [(13.5, 29.1), (23.3, 34.4)]$. On the other hand, figure 1(*b*) shows that the values of λ_4 and λ_5 are negative. They, also, have negative values of other system parameters, as we discuss below. Figure 1(*c*) shows the hyperchaotic attractors of (1.2) in (v_1, v_2, v_5) space for $a = 30, c = 90$ and $b = 11$. We calculate the Lyapunov exponents for this case and find $\lambda_1 = 5.333\,444, \lambda_2 = 0.282\,395, \lambda_3 = 0, \lambda_4 = -44.639\,339, \lambda_5 = -66.901\,647$.

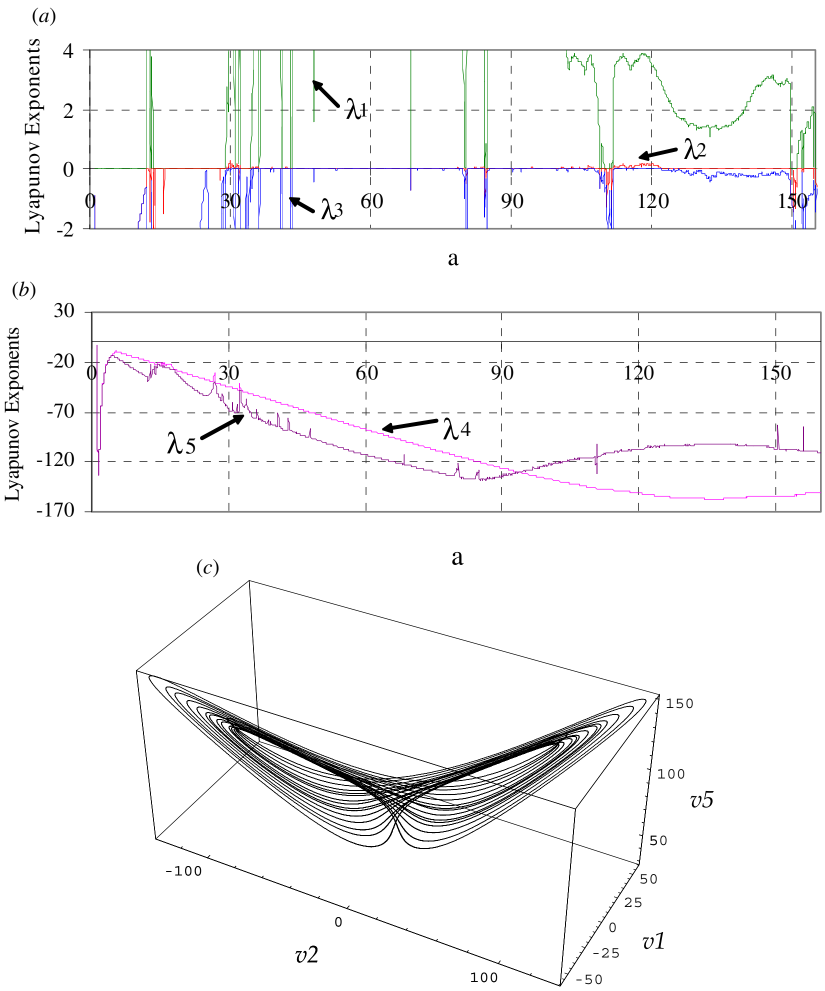

Figure 1. Lyapunov exponents equation (2.7) of (1.2) and its solutions at $b = 11$, $c = 90$ with $t_0 = 0$, $v_1(0) = 2$, $v_2(0) = 4$, $v_3(0) = 1$, $v_4(0) = 3$ and $v_5(0) = 2$. (*a*) λ_1, λ_2 and λ_3 versus a. (*b*) λ_4 and λ_5 versus a. (*c*) For $a = 30$, the hyperchaotic attractor of (1.2) in (v_1, v_2, v_5) space. (*d*) For $a = 47.9$, the chaotic attractor of (1.2) in (v_3, v_4, v_5) space.

Therefore, the Lyapunov dimension of this hyperchaotic attractor using equation (2.10) is $D \cong 3.1258$. Figure 1(*d*) shows the chaotic attractors of (1.2) in (v_3, v_4, v_5) space for $a = 47.9, c = 90$ and $b = 11$, for which the Lyapunov exponents are $\lambda_1 = 1.570\,294$, $\lambda_2 = 0, \lambda_3 = -0.447\,66, \lambda_4 = -70.208\,81, \lambda_5 = -87.282\,716$. Its Lyapunov dimension is approximately equal to 3.0159.

2.3.2. Fix $a = 30, c = 90$ and vary b. The values of λ_1, λ_2 and λ_3 versus b are plotted in figure 2(*a*) for the same initial conditions as in figure 1. From this figure we conclude that our system (1.2) has hyperchaotic attractors for $b \in (8.7, 11.9)$, chaotic attractors for $b \in [(0, 7.2), (20.6, 26.9)]$, periodic solutions for $b \in (18.7, 20.6)$, $(b \geqslant 26.9)$ and quasi-periodic solutions for $b \in [(7.2, 8.5), [12.9, 14.5), [14.6, 16)]$.

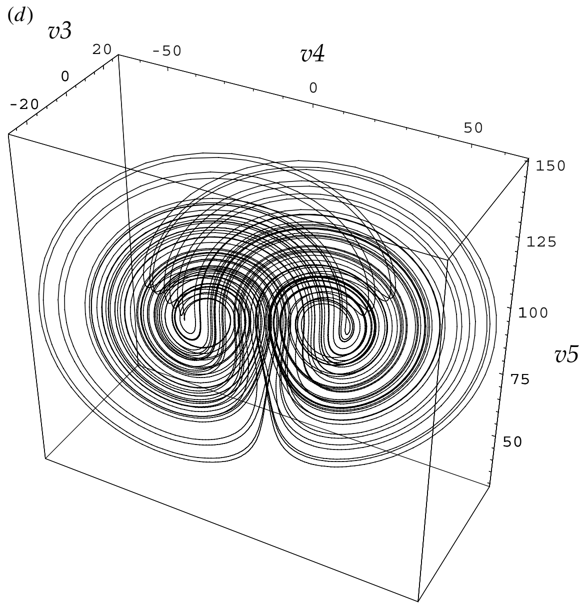

Figure 1. (Continued.)

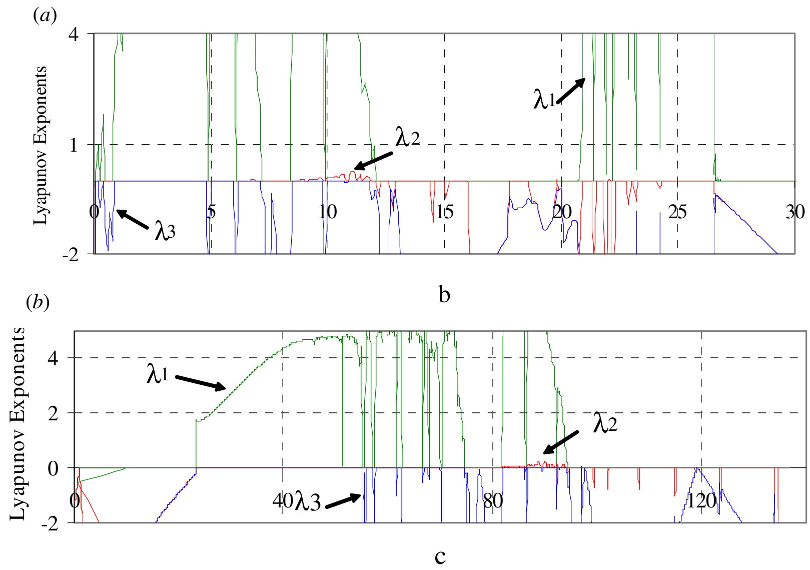

Figure 2. Lyapunov exponents of (1.2) with the same initial conditions as in figure 1: (*a*) λ_1, λ_2 and λ_3 versus b at $a = 30$, $c = 90$. (*b*) λ_1, λ_2 and λ_3 versus c at $a = 30$, $b = 11$.

2.3.3. Fix $a = 30, b = 11$ and vary c. As we did in (2.3.2) we plot λ_1, λ_2 and λ_3 versus c in figure 2(*b*). Figure 2(*b*) depicts that (1.2) has hyperchaotic attractors for $c \in [(81.8, 86.2), (86.5, 89.3), (89.6, 91.6)]$ and chaotic attractors exist for c lying in the intervals $[(23.3, 70), [70.6, 74.6)]$. Also for $c \in (1, 23.3)$ system (1.2) has periodic solutions, quasi-periodic solutions for $c \in [(74.6, 81.8), (94.7, 102), (102.3, 107.9)]$, $(c \geqslant 108.2)$ and the solutions of system (1.2) approach fixed points for $c \in (0, 1]$.

As is shown in figures 1(*a*) and 2, and based on signs of λ_1 and λ_2, our system (1.2) (or (2.1)) has different solutions for very small interval values of the parameters a, b and c. The dynamics of (1.2) is more rich in the sense that it exhibits both chaotic and hyperchaotic attractors and periodic and quasi-periodic attractors and solutions that approach fixed points.

3. Synchronization of hyperchaotic attractors of (1.2)

In this section, we apply the nonlinear control method [19, 20] to synchronize two identical hyperchaotic attractors of a complex nonlinear system (1.2). First, we point out the design of this method as follows.

3.1. Design of a controller via the nonlinear control method [19]

Consider the following system:

$$\dot{v}_d = Av_d + Bf(v_d), \tag{3.1}$$

where $v_d \in R^n$ is the state vector, $A \in R^{nn}, B \in R^n$ are matrix and vector of system parameters, respectively, and $f : R^n \to R^n$ is a nonlinear function. Equation (3.1) is considered as a drive system.

Inserting an additive controller $U \in R^n$, the controlled response system is then given by

$$\dot{v}_r = Av_r + Bf(v_r) + U, \tag{3.2}$$

where v_r denotes the state vector of the response system.

The synchronization problem is to design a controller U which synchronizes the states of both the drive and the response systems. We subtract (3.1) from (3.2) to get (3.3):

$$\dot{e_v} = A(v_r - v_d) + B[f(v_r) - f(v_d)] + U, \tag{3.3}$$

where $e_v = v_r - v_d$.

The essential purpose from the synchronization is to make $e_v(t)$ tend to zero as $t \to \infty$; therefore, we shall introduce a Lyapunov error function as $V = 1/2 \sum_{i=1}^{n} e_{vi}^2$, where V is obviously positive definite. Assuming that the parameters are known and the states are measurable, under a good choice of the controller U, which makes the first derivative of V as negative (i.e. $\dot{V} < 0$), we may be able to achieve synchronization. Consequently, the states of the response system and drive system will be globally synchronized asymptotically.

3.2. Synchronization of hyperchaotic attractors

Let us assume that we have two identical hyperchaotic attractors of system (1.2) and denote the drive system by the subscript d, while the response system to be controlled is denoted by the subscript r. Our aim is to design a controller U which will make the controlled response system follow the drive system and become ultimately the same; the drive and response systems are defined respectively as

$$\dot{x}_d = a(y_d - x_d) + y_d z_d, \qquad \dot{y}_d = cx_d - y_d - x_d z_d, \qquad \dot{z}_d = -bz_d + \tfrac{1}{2}(\bar{x}_d y_d + x_d \bar{y}_d) \tag{3.4}$$

and

$$\begin{aligned} &\dot{x}_r = a(y_r - x_r) + y_r z_r + (u_1 + iu_2), \qquad \dot{y}_r = cx_r - y_r - x_r z_r + (u_3 + iu_4), \\ &\dot{z}_r = -bz_r + \tfrac{1}{2}(\bar{x}_r y_r + x_r \bar{y}_r) + u_5, \end{aligned} \tag{3.5}$$

where a, b, c are positive (real or complex) parameters, x and y are complex variables (or functions), z is a real variable, the overbar denotes a complex conjugate variable and dots represent derivatives with respect to time, whereas $[u_1, u_2, u_3, u_4, u_5]$ are the control functions to be determined.

The complex systems (3.4) and (3.5) can be rewritten respectively as

$$\begin{aligned} &\dot{v}_{1d} = a(v_{3d} - v_{1d}) + v_{3d}v_{5d}, && \dot{v}_{2d} = a(v_{4d} - v_{2d}) + v_{4d}v_{5d}, \\ &\dot{v}_{3d} = cv_{1d} - v_{3d} - v_{1d}v_{5d}, && \dot{v}_{4d} = cv_{2d} - v_{4d} - v_{2d}v_{5d}, \\ &\dot{v}_{5d} = -bv_{5d} + (v_{1d}v_{3d} + v_{2d}v_{4d}) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\dot{v}_{1r} = a(v_{3r} - v_{1r})v_{3r}v_{5r} + u_1, && \dot{v}_{2r} = a(v_{4r} - v_{2r}) + v_{4r}v_{5r} + u_2, \\ &\dot{v}_{3r} = cv_{1r} - v_{3r} - v_{1r}v_{5r} + u_3, && \dot{v}_{4r} = cv_{2r} - v_{4r} - v_{2r}v_{5r} + u_4, \\ &\dot{v}_{5r} = -bv_{5r} + (v_{1r}v_{3r} + v_{2r}v_{4r}) + u_5. \end{aligned} \tag{3.7}$$

In order to now obtain the controller $U = [u_1, u_2, u_3, u_4, u_5]^T$, we define the error states between the response system that is to be controlled and the controlling drive system as

$$\begin{aligned} &e_{v_1} = (v_{1r} - v_{1d}), && e_{v_2} = (v_{2r} - v_{2d)}), && e_{v_3} = (v_{3r} - v_{3d}), \\ &e_{v_4} = (v_{4r} - v_{4d)}), && e_{v_5} = (v_{5r} - v_{5d}). \end{aligned} \tag{3.8}$$

Subtracting (3.6) from (3.7) and using (3.8) yield the error equations:

$$\begin{aligned} \dot{e}_{v_1} &= -ae_{v_1} + ae_{v_3} + v_{3r}e_{v_5} + v_{5d}e_{v_3} + u_1, \\ \dot{e}_{v_2} &= -ae_{v_2} + ae_{v_4} + v_{4r}e_{v_5} + v_{5d}e_{v_4} + u_2, \\ \dot{e}_{v_3} &= ce_{v_1} - e_{v_3} - v_{1r}e_{v_5} - v_{5d}e_{v_1} + u_3, \\ \dot{e}_{v_4} &= ce_{v_2} - e_{v_4} - v_{2r}e_{v_5} - v_{5d}e_{v_2} + u_4, \\ \dot{e}_{v_5} &= -be_{v_5} + v_{1r}e_{v_3} + v_{3d}e_{v_1} + v_{2r}e_{v_4} + v_{4d}e_{v_2} + u_5. \end{aligned} \tag{3.9}$$

For positive parameters a, b and c, we may define a Lyapunov function for equation (3.9) by the following quantity:

$$V(t) = \frac{1}{2}\sum_{i=1}^{5} e_{v_i}^2. \tag{3.10}$$

The first derivative of $V(t)$ is given by

$$\begin{aligned} \dot{V}(t) = {} & ae_{v_1}e_{v_3} - ae_{v_1}^2 + v_{3r}e_{v_5}e_{v_1} + v_{5d}e_{v_3}e_{v_1} + u_1e_{v_1} + ae_{v_2}e_{v_4} - ae_{v_2}^2 + v_{4r}e_{v_5}e_{v_2} \\ & + v_{5d}e_{v_4}e_{v_2} + u_2e_{v_2} + ce_{v_1}e_{v_3} - e_{v_3}^2 - v_{1r}e_{v_5}e_{v_3} - v_{5d}e_{v_3}e_{v_1} + u_3e_{v_3} + ce_{v_2}e_{v_4} \\ & - e_{v_4}^2 - v_{2r}e_{v_5}e_{v_4} - v_{5d}e_{v_4}e_{v_2} + u_4e_{v_4} - be_{v_5}^2 + v_{1r}e_{v_5}e_{v_3} + v_{2r}e_{v_5}e_{v_4} \\ & + v_{3d}e_{v_5}e_{v_1} + v_{4d}e_{v_5}e_{v_4} + u_5e_{v_5}. \end{aligned} \tag{3.11}$$

From (3.11), we have

$$\begin{aligned} \dot{V}(t) = {} & -a\left(e_{v_1}^2 + e_{v_2}^2\right) - \left(e_{v_3}^2 + e_{v_4}^2\right) - be_{v_5}^2 + (u_1 + ae_{v_3} + v_{3d}e_{v_5} + v_{3r}e_{v_5})e_{v_1} \\ & + (u_2 + ae_{v_4} + v_{4d}e_{v_5} + v_{4r}e_{v_5})e_{v_2} + (u_3 + ce_{v_1} + v_{1r}e_{v_5})e_{v_3} + (u_4 + ce_{v_2} \\ & + v_{2r}e_{v_5})e_{v_4} + (u_5 + v_{1r}e_{v_3} + v_{2r}e_{v_4})e_{v_5}. \end{aligned} \tag{3.12}$$

There are many possible choices for the controller U. If we choose the control input function u_i as

$$\begin{aligned} &u_1 = -(ae_{v_3} + v_{3d}e_{v_5} + v_{3r}e_{v_5}), && u_2 = -(ae_{v_4} + v_{4d}e_{v_5} + v_{4r}e_{v_5}), \\ &u_3 = -ce_{v_1} + v_{1r}e_{v_5}, && u_4 = -ce_{v_2} + v_{2r}e_{v_5} \quad \text{and} \\ &u_5 = -(v_{1r}e_{v_3} + v_{2r}e_{v_4}), \end{aligned} \tag{3.13}$$

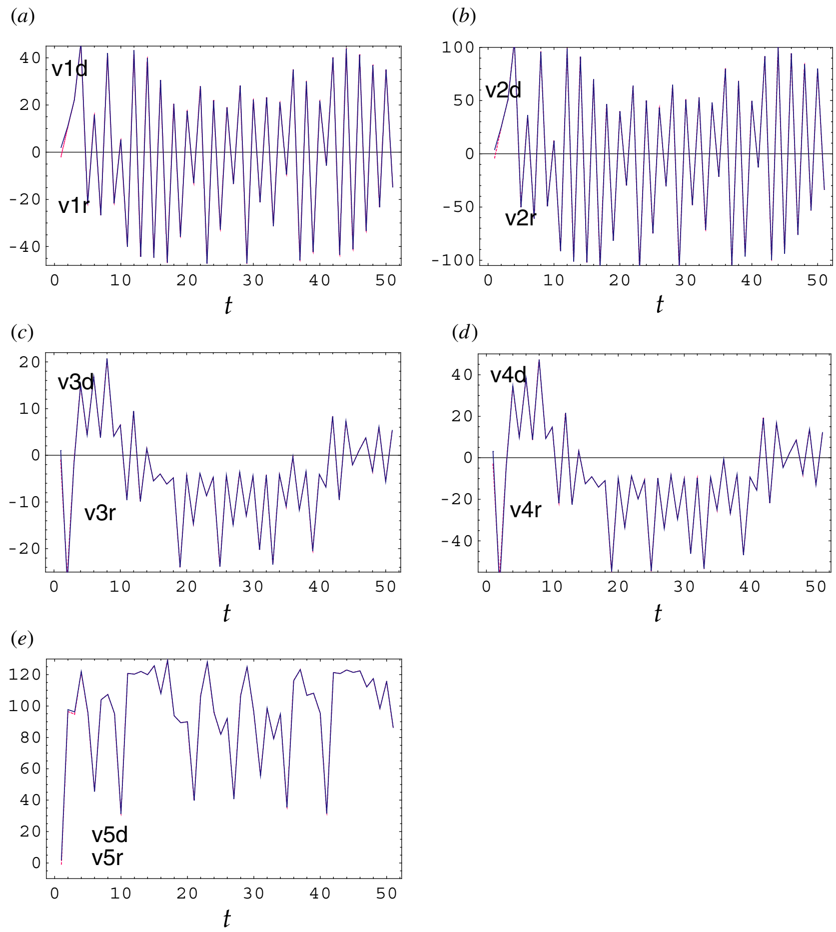

Figure 3. Hyperchaotic synchronization of systems (3.6) and (3.7) for $a = 30, c = 90$ and $b = 11$ with $t_0 = 0, v_{1d}(0) = 2, v_{2d}(0) = 4, v_{3d}(0) = 1, v_{4d}(0) = 3, v_{5d}(0) = 2$ and $v_{1r}(0) = -2, v_{2r}(0) = -4, v_{3r}(0) = -1, v_{4r}(0) = -3, v_{5r}(0) = -2$. (*a*) $v_{1d}(t)$ and $v_{1r}(t)$ versus t, (*b*) $v_{2d}(t)$ and $v_{2r}(t)$ versus t, (*c*) $v_{3d}(t)$ and $v_{3r}(t)$ versus t, (*d*) $v_{4d}(t)$ and $v_{4r}(t)$ versus t, (*e*) $v_{5d}(t)$ and $v_{5r}(t)$ versus t (t = time/10).

equation (3.12) becomes

$$\dot{V}(t) = -a\left(e_{v_1}^2 + e_{v_2}^2\right) - \left(e_{v_3}^2 + e_{v_4}^2\right) - be_{v_5}^2 < 0. \tag{3.14}$$

Since $V(t)$ is a positive definite function and its derivative is negative definite, then based on the Lyapunov stability theory, the error states $e_{v_i} = 0, i = 1, \ldots, 5$, are asymptotically stable, which means that

$$\lim_{t \longrightarrow \infty} \|e_{v_i}(t)\| = 0.$$

Therefore, the states of controlled response and drive systems are globally synchronized asymptotically. Systems (3.6) and (3.7) with (3.13) are solved numerically for $a = 30$, $c = 90$ and $b = 11$ and initial conditions of the drive and response systems at $t_0 = 0$ are $v_{1d}(0) = 2$, $v_{2d}(0) = 4$, $v_{3d}(0) = 1$, $v_{4d}(0) = 3$, $v_{5d}(0) = 2$ and $v_{1r}(0) = -2$, $v_{2r}(0) = -4$, $v_{3r}(0) = -1$, $v_{4r}(0) = -3$, $v_{5r}(0) = -2$. The synchronization of this hyperchaotic attractor is shown in figure 3, where the oscillations of the drive and response systems rapidly become

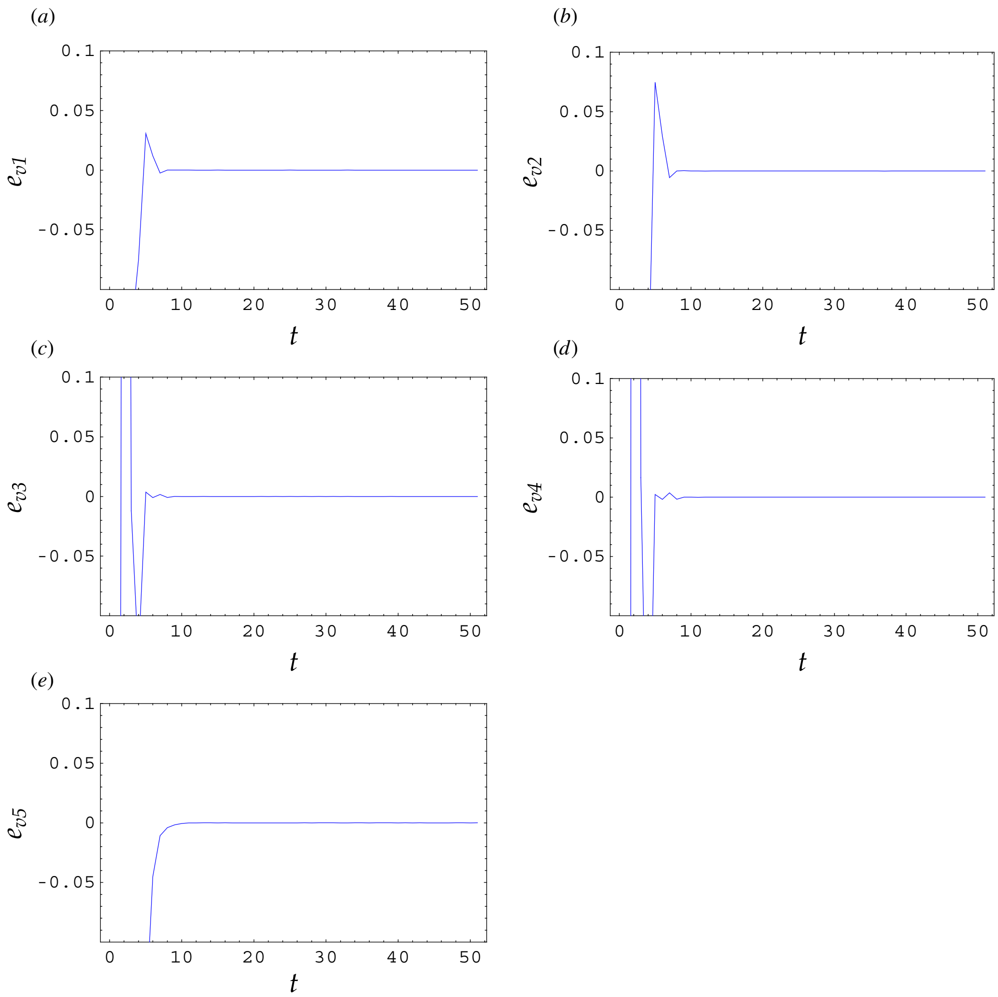

Figure 4. Synchronization errors (solutions of system (3.9)). (*a*) (e_{v_1}, t) diagram, (*b*) (e_{v_2}, t) diagram, (*c*) (e_{v_3}, t) diagram, (*d*) (e_{v_4}, t) diagram, (*e*) (e_{v_5}, t) diagram.

totally indistinguishable. The synchronization errors, e_{v_i}, plotted in figure 4, also demonstrate that synchronization is achieved very fast, as they are seen to converge to zero after very small values of t (time/10).

Finally, we point out that very similar results are obtained, when we apply the above technique of nonlinear control to synchronize two identical *chaotic* attractors of system (1.2). Entirely analogous figures are obtained for the oscillations as well as the behavior of the errors, which we do not display here as they are qualitatively the same as figures 3 and 4.

4. Conclusions

As is well known, there exist interesting cases of dynamical systems where the main variables participating in the dynamics are complex, for example, when amplitudes of electromagnetic fields and atomic polarization are involved [2]. Increasing the number of variables (or introducing complex variables) is also crucial in chaos synchronization used in secure communications, where one wishes to maximize the content and security of the transmitted information.

In this paper, we have analyzed a new complex nonlinear hyperchaotic system, represented by five first-order nonlinear ordinary differential equations. Dynamical properties such as dissipative behavior, fixed points and their stability, chaotic and hyperchaotic attractors were studied.

Based on the computation of the spectrum of the Lyapunov exponents, the fractional Lyapunov dimension was computed for these attractors and was found in all cases to be greater than 3, with the dimension of the hyperchaotic attractors being somewhat larger than that of the chaotic ones. Compared with the complex Lorenz system and the complex Chen and Lü systems, our model was seen to possess certain distinct differences such as having *two* circles of equilibria and large parameter intervals of hyperchaotic and chaotic behavior. On the other hand, when applying the same nonlinear control method, we discovered that both chaotic and hyperchaotic attractors are synchronized very rapidly and in a very similar way.

It is hoped that the results reported here increase our knowledge of the dynamics of complex dynamical systems, which is still far from what has been achieved to date for real dynamical systems.

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References

[1] Fowler A C, McGuinnes M J and Gibbon J D 1982 The complex Lorenz equations *Physica* D **4** 139–63

[2] Rauh A, Hannibal L and Abraham N B 1996 Global stability properties of the complex Lorenz model *Physica* D **99** 45–58

[3] Mahmoud G M, Al-Kashif M A and Aly S A 2007 Basic properties and chaotic synchronization of complex Lorenz system *Int. J. Mod. Phys.* C **18** 253–65

[4] Qi G, Chen G, Du S, Chen Z and Yuan Z 2005 Analysis of a new chaotic system *Physica* A **352** 295–308

[5] Chen G 1999 *Controlling Chaos and Bifurcations in Engineering Systems* (Boca Raton, FL: CRC Press)

[6] Peng J H, Ding E J, Ding M and Yang W 1996 Synchronizing hyperchaos with a scalar transmitted signal *Phys. Rev. Lett.* **76** 904–7

[7] Mahmoud G M, Bountis T and Mahmoud E E 2007 Active control and global synchronization of complex Chen and Lü systems *Int. J. Bifurcation and Chaos* **17**

[8] Mahmoud G M, Aly S A and Al-Kashif M A 2008 Dynamical properties and chaos synchronization of a new chaotic complex nonlinear system *Nonlinear Dyn.* **51** 171–81

[9] Mahmoud G M, Aly S A and Farghaly A A 2007 On chaos synchronization of a complex two coupled dynamos system *Chaos Solitons Fractals* **33** 178–87

[10] Mahmoud G M and Bountis T 2004 The dynamics of systems of complex nonlinear oscillators: a review *Int. J. Bifurcation Chaos* **14** 3821–46

[11] Ning C Z and Haken H 1990 Detuned lasers and the complex Lorenz equations—subcritical and supercritical Hopf bifurcations *Phys. Rev.* A **41** 3827–3837

[12] Fowler A C, Gibbon J D and McGuinnes M J 1983 The real and complex Lorenz equations and their relevance to physical systems *Physica* D **7** 126–34

[13] Kiselev A D 1998 Symmetry breaking and bifurcations in complex Lorenz model *J. Phys. Stud.* **2** 30–7

[14] Rössler O E 1979 An equation for hyperchaos *Phys. Lett.* A **71** 155–7

[15] Chen A, Lu J A, Lü J and Yu S 2006 Generating hyperchaotic Lü attractor via state feedback control *Physica* A **364** 103–10

[16] Haeri M and Dehghaui M 2006 Impulsive synchronization of Chen's hyperchaotic system *Phys. Lett.* A **356** 226–30

[17] Frederickson P, Kaplan J L, Yorke E D and Yorke J A 1983 The Liapunov dimension of strange attractors *J. Diff. Eqns* **44** 185–207

[18] Shuster H G 1982 *Deterministic Chaos* (Berlin: Springer)

[19] Huang L, Feng R and Wang M 2004 Synchronization of chaotic systems via nonlinear control *Phys. Lett.* A **320** 271–5

[20] Park J H 2005 Chaos synchronization of a chaotic system via nonlinear control *Chaos Solitons Fractals* **25** 699–704

[21] Wolf A, Swift J, Swinney H and Vastano J 1985 Determining Lyapunov exponents from a time series *Physica* D **16** 285–317